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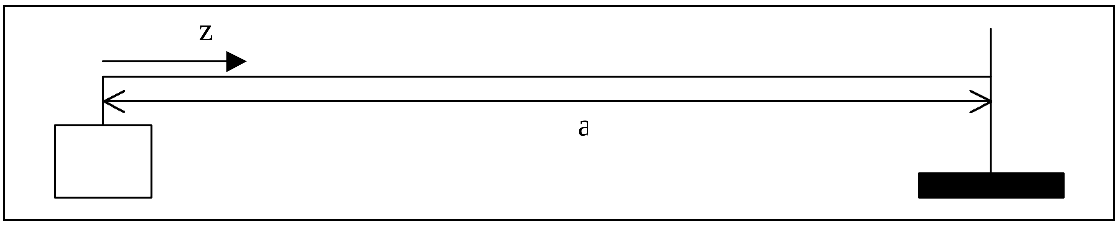

Let the upward displacement of the rope at the displacement z from left end be $y\ (z,t)$.

It satisfies the wave equation with damping b:

$$\frac{\partial^2 y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{2b}{c}\frac{\partial y}{\partial t} = 0 \qquad \ldots\ldots\ldots(1)$$

The two boundary conditions are

1. oscillation of the vibrator at the left end: $y\ (0,t) = Ae^{iwt}$
2. the right end is fixed: $y\ (a,t) = 0$

We introduce the conversion

$$y(z,t) = y\ (z,t) - (1 - \frac{z}{a})Ae^{iwt} \qquad \ldots\ldots\ldots(2)$$

Hence, (1) becomes

$$\frac{\partial^2 y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{2b}{c}\frac{\partial y}{\partial t} = -A(1 - \frac{z}{a})(k^2 - 2ikb)e^{iwt} \qquad \ldots\ldots(3)$$

where $k = \dfrac{w}{c}$.

The boundary conditions of y are

$$y(a,t)\ =\ y(0,t)\ =\ 0$$

The problem now becomes a standard forced oscillation on a string with its two ends fixed. Its solutions can be found in many textbooks of intermediate mechanics (e.g.Walter Hauser's *Introduction to the principles of Mechanics*, Addison-Wesley, 1965).

The solution is expressed as a sum of the normal modes,

$$y(z,t) = \frac{2A}{p}\sum_{n=1}^{\infty}\{ \frac{k^2(k_{\mathrm{n}}^2 - k^2 - 4b^2)}{n[(k_n^2 - k^2)^2 + 4k^2b^2]}\cos wt + \frac{2bkk_n^2}{n[(k_n^2 - k^2)^2 + 4k^2b^2]}\sin wt\}\sin k_n z$$

where $k_n = \frac{np}{a}$.

It is a resonance response when $k = k_n$

By putting $y(z,t)$ back into (2), we get $y\,(z,t)$.