Energy in a String Wave

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When one end of a taut horizontal elastic string is shaken repeatedly up and down, a transverse wave (assume sine waveform) will be produced and travel along it.[1] College students know this type of wave motion well. They know when the wave passes by, each element of the string will perform an oscillating up-down motion, which in mechanics is termed *simple harmonic*.[2] They also know elements of the string at the highest and the lowest positions—the crests and the troughs—are momentarily at rest, while those at the centerline (zero displacement) have the greatest speed, as shown in Fig. 1. Irrespective of this, they are less familiar with the energy associated with the wave. They may fail to answer a question such as, "In a traveling string wave, which elements have respectively the greatest kinetic energy (KE) and the greatest potential energy (PE)?" The answer to the former is not difficult; elements at zero position have the fastest speed and hence their KE, being proportional to the square of speed, is the greatest. To the PE, what immediately comes to their mind may be the simple harmonic motion (SHM), in which the PE is the greatest and the KE is zero at the two turning points. It may thus lead them to think elements at crests or troughs have the greatest PE. Unfortunately, this association is wrong. Thinking that the crests or troughs have the greatest PE is a misconception.[3]

What kind of PE?

First of all, we need to clarify what kind of PE we are talking about. It is not the gravitational PE, *mgh*, because the weight (not the mass) of the string does not play a role in the propagation of the wave. If the weight's effect is taken into account, the waveform will no longer be up-down symmetrical. Usually, the gravitational effect is neglected although it may not be stated explicitly. The relevant PE must be the elastic PE (or called *strain energy*) that is involved because the string is inevitably stretched when a transverse wave moves along it. To accommodate a wave, a straight string has to be curved and hence its length must be increased.

Max PE occurs at max KE

Figure 2 tells the whole story. It shows the shape of a portion of the string at three instants. Because wave motion does not transport any matter, the two labeled elements in Fig. 2, P and Q, only move vertically up and down. The waveform varies with time and so does the distance *PQ*. The separation between P and Q is the greatest at the central position and the smallest at the crest or trough. In other words, the string is not evenly stretched; the elements at the central position are stretched more than those at the crests or troughs.[4,5] Therefore, the elastic PE, being proportional to the square of extension, is the greatest for those elements at zero position. Advanced derivation from the first principles reveals the elastic PE in a string wave is proportional to the square of the slope of the waveform, not to the square of the displacement from the equilibrium position.[6] Now, we see an important fact that the KE and PE of an element (technically, they are called *energy densities*) rise and fall together; they reach their maximum values simultaneously at zero position and minimum values at the crest or trough.[7] A direct implication of this is the total mechanical energy, i.e., the sum of PE and KE, of a specific element is not a constant. There is no surprise because the action of a traveling wave is to transport energy; each element cannot be a closed system. Integrating each element's PE over a complete wavelength is found to be proportional to the square of amplitude, like the maximum PE in SHM. Nevertheless, the PE in a string wave does not have its maximum at

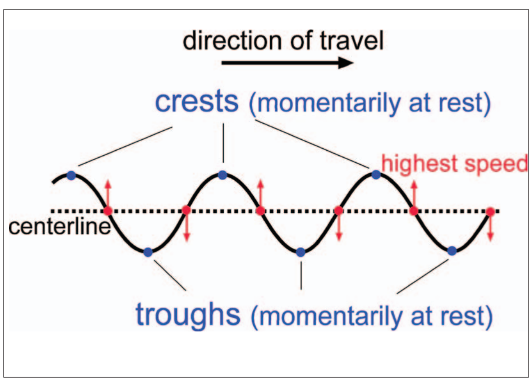

Fig. 1. A transverse traveling wave.

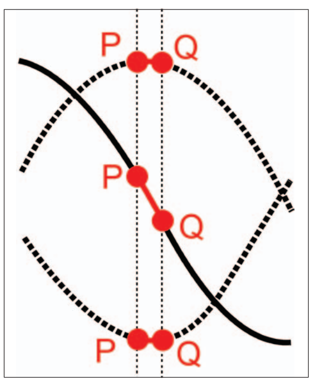

Fig. 2. Elements only move up and down. The PQ element is stretched the most at the zero position.

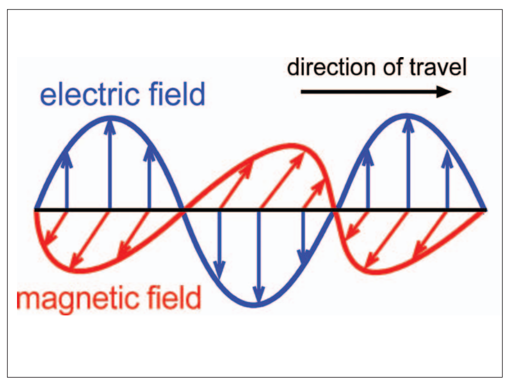

Fig. 3. An electromagnetic traveling wave.

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the amplitude, a point of which we should be aware.

It is worthwhile to note that string wave is not an exceptional example of such synchronous variation of energies. This property is common to other traveling waves regardless of whether they are mechanical or electromagnetic, transverse or longitudinal. For instance, the energy carried by an electromagnetic traveling wave is stored in the in-phase vibrating electric field and the magnetic field of the wave, as shown in Fig. 3. Being proportional to the square of the corresponding field strength, the energies associated with these two kinds of fields rise and fall together. In a longitudinal traveling wave, the centers of compressions and rarefactions have the greatest PE and the greatest KE too.

The discussion presented above is inapplicable to a standing wave, in which the waveform does not advance and so there is no transfer of energy. In each period the curved string becomes straight twice. At the instants when all elements have their maximum displacement simultaneously, all elements are stationary and the nodes have the greatest PE. When the string becomes straight, PE now transforms to KE and the antinodes have the greatest value. At any time the nodes of a standing wave do not possess any KE because they are always stationary. Thus, in a standing wave, the maximum PE and maximum KE occur at different times at different locations.

Acknowledgment

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References

1. Any realistic string must be extensible (elastic) although the extension may be extremely small. This requirement is essential in our model because we are going to show the string is extended by different amounts at different locations.
2. The small amplitude approximation is adopted. Under that model, the motion of each element is purely vertical. Mathematically, the horizontal component of tension is $T_x = T_0 \cos\theta$, where T_0 is the undisturbed tension (see Ref. 5) and θ is the angle between the horizontal and the string. Using the identities $\cos\theta = 1/\sqrt{1+\tan^2\theta}$ and $\tan\theta = \partial y/\partial x$, we obtain $T_x = T_0 - (T_0/2)(\partial y/\partial x)^2 + \ldots$. Obviously, T_x is not constant along the string unless the second and the higher order terms are dropped (the small amplitude approximation).
3. Seems common. See, for example, http://hyperphysics.phy-astr.gsu.edu/hbase/waves/powstr.html and http://www.physics.upenn.edu/courses/gladney/phys151/lectures/lecture_mar_31_2003.shtml.
4. D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 7th ed. (Wiley, 2004), p. 423. Few popular textbooks on general physics have and depict correctly the PE variation in a string wave like this.
5. Let $T = T_0 + T_1$, where T_0 is the tension in the string before the propagation of the wave (called "undisturbed tension") and T_1 is the part contributed from the uneven extension of the string. Because T_1 itself is a term proportional to the square of the slope of the waveform (see Ref. 6), so each element only moves vertically if the small amplitude approximation is still assumed.
6. See, for examples, http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/AnalyzingWaves.htm; H.Benson, *University Physics*, rev. ed. (Wiley, 2006), p. 337; and W. N. Mathews, Jr., "Energy in a one-dimensional small amplitude mechanical wave," *Am. J. Phys.* **53,** 974–978 (Oct. 1985).
7. As defined in Ref. 5, the tension is split into two parts, T_0 and T_1. Each element of the string always possesses the same amount of PE corresponding to T_0, but the wave does not transfer this kind of PE. What this paper focuses on is the PE corresponding to T_1, which is found to be position-dependent. Hence, at the crests or troughs, the PE is minimum but not zero.

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Exponential

In a recent decision about the Navy's use of high-power sonar (underwater sound used for the detection of submarines) and the effect these large amplitude acoustical signals may have on marine mammals, the U.S. Supreme Court ruled in favor of the Navy. In a news story about the decision we read: "Chief Justice Roberts took issue with both restrictions. The Navy had agreed to shut down its sonar if marine mammals were sighted within 200 yards. The appeals court's requirement that it increase the zone to 2,200 yards, Chief Justice Roberts said, would 'expand the surface area of the shut-down zone by a factor of over 100,' given 'the exponential relationship between radius length and surface area.'"[1]

1. A. Liptak, *New York Times*, (Web Version) November 13, 2008.